

Mathematics Framework Solution Sets

Grade 6

A note about these solutions.

These solutions are intended for teachers, not students. The solutions are fairly detailed and some include additional comments that serve to further explain the content and purpose of each problem. It is important to note that these solutions are not meant to be representative of student solutions.

It is the nature of many mathematics problems that they can be solved in different ways. The solutions given here represent simply one way of solving the problems. At times, a second solution path is offered in the Further Explanation boxes.

It is our hope that these solution sets help teachers to better see the essential skills and concepts that are important to student success in Grade 6 mathematics.

Number Sense 1.2 Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations (a/b , a to b , $a : b$).

Number Sense 1.3 Use proportions to solve problems (e.g., determine the value of N if $4/7 = N/21$, find the length of a side of a polygon similar to a known polygon). Use cross multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

Problem: Complete the following statements:

1. If $3 \text{ ft} = 1 \text{ yd}$, then $7 \text{ ft} = ? \text{ yd}$.
2. If $32 \text{ oz} = 1 \text{ qt}$, then $6.7 \text{ qt} = ? \text{ oz}$.

Solution: 1. We set up the proportion

$$\frac{3 \text{ ft}}{1 \text{ yd}} = \frac{7 \text{ ft}}{x \text{ yd}},$$

where x represents the unknown number of yards. This can be seen as the equivalence of two quantities:

$$3 \text{ ft/yd} = \frac{7}{x} \text{ ft/yd},$$

which leads to solving the simple proportion

$$3 = \frac{7}{x}.$$

Equivalently,

$$\frac{1}{3} = \frac{x}{7} \Rightarrow x = \frac{7}{3}.$$

Therefore

$$7 \text{ ft} = \frac{7}{3} \text{ yd} = 2\frac{1}{3} \text{ yd}.$$

2. Again we set up the proportion

$$\frac{32 \text{ oz}}{1 \text{ qt}} = \frac{x \text{ oz}}{6.7 \text{ qt}}.$$

We solve the numerical proportion by cross multiplying:

$$\frac{32}{1} = \frac{x}{6.7} \Rightarrow 32 \times 6.7 = x \times 1$$

Therefore $x = 214.4$, so that $6.7 \text{ qt} = 214.4 \text{ oz}$.

Number Sense 1.2 Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations (a/b , a to b , $a : b$).

Number Sense 1.3 Use proportions to solve problems (e.g., determine the value of N if $4/7 = N/21$, find the length of a side of a polygon similar to a known polygon). Use cross multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

Problem: In a lemonade punch, the ratio of lemonade to soda pop is 2 : 3. If there are 24 gallons of punch, how much lemonade is needed?

Solution: The ratio of 2 : 3 indicates that for every 2 parts of lemonade we require 3 parts of soda pop. This means that any mixture of the punch consists of $2x$ gallons of lemonade to $3x$ gallons of soda pop. If the total amount of punch is 24 gallons, then we can solve:

$$2x + 3x = 24 \quad \Rightarrow \quad 5x = 24 \quad \Rightarrow \quad x = \frac{24}{5} = 4\frac{4}{5} \text{ gal.}$$

Thus, $4\frac{4}{5}$ can be considered as the “part” of lemonade or soda pop. To find the amount of lemonade in the punch, we find

$$\text{amount of lemonade} = 2x \text{ gal} = 2 \left(4\frac{4}{5} \right) \text{ gal} = 8\frac{8}{5} \text{ gal} = 9\frac{3}{5} \text{ gal.}$$

Number Sense 2.1 Solve problems involving addition, subtraction, multiplication, and division of positive fractions and explain why a particular operation was used for a given situation.

Problem: Find the sum $\frac{5}{6} + \frac{3}{10}$.

Solution: The least common denominator of the fractions is the least common multiple (LCM) of the numbers 6 and 10, which is 30. Thus,

$$\begin{aligned}\frac{5}{6} + \frac{3}{10} &= \frac{5 \times 5}{6 \times 5} + \frac{3 \times 3}{10 \times 3} \\ &= \frac{25}{30} + \frac{9}{30} \\ &= \frac{34}{30}.\end{aligned}$$

We can divide numerator and denominator by 2, since both are even, to reduce the fraction to $\frac{17}{15}$. As a mixed number, the sum is $1\frac{2}{15}$.

Further Explanation: While the method of finding the least common denominator for adding and subtracting fractions is frequently employed, it should be noted that these operations can always be performed by the simple method of first finding

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd},$$

and then reducing the answer.

For example, the above computation could be performed in the following way:

$$\frac{5}{6} + \frac{3}{10} = \frac{5 \cdot 10 + 6 \cdot 3}{6 \cdot 10} = \frac{50 + 18}{60} = \frac{68}{60},$$

which reduces to $\frac{17}{15}$.

Number Sense 2.3 Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations that use positive and negative integers and combinations of these operations.

Number Sense 2.4 Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).

Problem: Write the following as an integer over a whole number:

$$8, -6, 4\frac{1}{2}, -1\frac{1}{5}, 0, 0.013, -1.5.$$

Solution: We write each as a fraction of the form $\frac{a}{b}$ for a an integer and b a whole number:

$$\begin{aligned}8 &= \frac{8}{1} \\-6 &= \frac{-6}{1} \\4\frac{1}{2} &= 4 + \frac{1}{2} = \frac{8}{2} + \frac{1}{2} = \frac{9}{2} \\-1\frac{1}{5} &= -1 + \frac{1}{5} = -\frac{5}{5} + \frac{1}{5} = \frac{-6}{5} \\0 &= \frac{0}{1} \\0.013 &= \frac{13}{1000} \\-1.5 &= \frac{-3}{2}\end{aligned}$$

Further Explanation: Of course, there is more than one way to write each of these numbers as a fraction of the form $\frac{a}{b}$ with a an integer and b a whole number. For example,

$$8 = \frac{8}{1} = \frac{16}{2} = \frac{24}{3} = \dots$$

Number Sense 2.4 Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).

Problem:

1. Find the least common multiple of 6 and 10 (count by sixes until you come to a multiple of 10).
2. List the first 20 multiples of 6.
3. List the first 20 multiples of 10.
4. List all the multiples that 6 and 10 have in common that are less than or equal to 120.

Solution:

1. If we count by sixes, we reach 30 as the least common multiple: 6, 12, 18, 24, 30, It is the smallest positive number that is a multiple of both 6 and 10.
2. The first 20 multiples of 6 are
 $6, 12, 18, 24, \underline{30}, 36, 42, 48, 54, \underline{60}, 66, 72, 78, 84, \underline{90}, 96, 102, 108, 114, \underline{120}.$
3. The first 20 multiples of 10 are
 $10, 20, \underline{30}, 40, 50, \underline{60}, 70, 80, \underline{90}, 100, 110, \underline{120}, 130, 140, 150, 160, 170, 180, 190, 200.$
3. The common multiples of 6 and 10 that are less than or equal to 120 are
 $30, 60, 90, 120.$

Number Sense 2.4 Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).

Problem:

1. Make a sieve of Eratosthenes up to 100.
2. Find the greatest common factor of 18 and 30 (list all factors of 18 until you come to a factor of 30).
3. Reduce $\frac{18}{30}$.

Solution: 1. We begin by creating a table of the numbers from 1 through 100. Then, we circle the first prime 2 and then cross off every other number after 2, removing the multiples of 2. Then we repeat this process with 3, removing every third number, the multiples of 3. As we continue this process, we reveal the primes between 1 and 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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2. The factors of 18 can be found by looking at factor pairs for 18 and listing them. These are (1, 18), (2, 9), and (3, 6), so that the factors of 18 are 1, 2, 3, 6, 9 and 18. Of these, the largest that is also a factor of 30 is 6.

3. To reduce the fraction $\frac{18}{30}$, we factor 18 and 30 using the greatest common factor: $18 = 6 \cdot 3$ and $30 = 6 \cdot 5$, and then write

$$\frac{18}{30} = \frac{6 \cdot 3}{6 \cdot 5} = \frac{3}{5}.$$

Algebra and Functions 1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.

Problem: Moe was paid \$7 per hour and earned \$80.50. How many hours did Moe work?

Solution: Suppose Moe worked for x hours. Then we can set up an equation relating the number of hours he worked and how much he was paid as follows:

$$7 \frac{\text{dollars}}{\text{hour}} \cdot x \text{ hours} = 80.5 \text{ dollars}$$

The units of hours cancel, and we obtain

$$7x \text{ dollars} = 80.5 \text{ dollars},$$

so that

$$7x = 80.5 \quad \Rightarrow \quad x = \frac{80.5}{7} = 11.5.$$

Therefore Moe worked for 11.5 hours.

Algebra and Functions 1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.

Problem: Write the following in symbolic notation using n to represent the number:

1. A number increased by 33.
2. The product of a number and (-7) .
3. $8\frac{1}{2}$ decreased by some number.
4. The square of some number which is then divided by 7.
5. The sum of some number and $\frac{1}{3}$ which is then increased by the third power of the same number

Solution:

1. “Increased” refers to addition: $n + 33$.

2. “Product” refers to the result of multiplication:

$$(-7) \times n = -7n.$$

3. “Decreased” refers to subtraction. In this case we are subtracting n from $8\frac{1}{2}$:

$$8\frac{1}{2} - n.$$

4. “Square” refers to multiplying a number by itself, so we have

$$n^2 \div 7 = \frac{n^2}{7}.$$

5. Here we have

$$n + \frac{1}{3} + n^3 = n^3 + n + \frac{1}{3}.$$

Algebra and Functions 1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.

Algebra and Functions 3.1 Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = 1/2bh$, $C = \pi d$ —the formulas for the perimeter of a rectangle, the area of a triangle, and the circumference of a circle, respectively).

Problem: A rectangle is constructed with 8 feet of string. Suppose that one side is $1\frac{4}{14}$ feet long. What is the length of the other side?

Solution: Since the rectangle was constructed with 8 feet of string, its perimeter is 8 feet. Since one side is $1\frac{4}{14}$ feet long, the opposite side is the same length, so the other pair of opposite sides makes up

$$\begin{aligned}8 - 2 \cdot 1\frac{4}{14} &= 8 - 2\frac{4}{7} \\&= 7\frac{7}{7} - 2\frac{4}{7} \\&= 5\frac{3}{7}\end{aligned}$$

feet of string.

Therefore, these sides have length

$$5\frac{3}{7} \div 2 = \frac{38}{7} \times \frac{1}{2} = \frac{19}{7},$$

or $2\frac{5}{7}$ feet.

Further Explanation: Another approach uses the perimeter formula for a rectangle, $P = 2l + 2w$. In this case, $P = 8$, so that

$$2(l + w) = 8 \quad \Rightarrow \quad l + w = 4.$$

Then, knowing that one side (the length, say) is $1\frac{4}{14}$ feet, we can solve

$$1\frac{4}{14} + w = 4$$

for w to find the length of the other side.

Algebra and Functions 1.3 Apply algebraic order of operations and the commutative, associative, and distributive properties to evaluate expressions; and justify each step in the process.

Problem: True or False?

$$(25 + 16) \times 6 = 25 + 16 \times 6.$$

Solution: This is false, since

$$(25 + 16) \times 6 = 41 \times 6 = 246,$$

while

$$25 + 16 \times 6 = 25 + 96 = 121.$$

The error was made by not properly distributing in the expression $(25 + 16) \times 6$:

$$\begin{aligned}(25 + 16) \times 6 &= 25 \times 6 + 16 \times 6 \\ &= 150 + 96 = 246.\end{aligned}$$

Further Explanation: Understanding the distributive property:

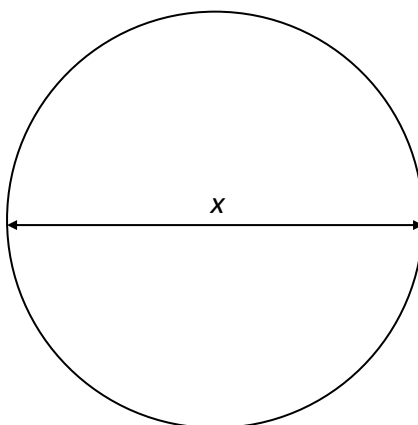
$$a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc$$

is crucial for further success in algebra and other areas of mathematics. It and other fundamental properties of numbers form the basis for working with variable expressions.

In addition, order of operations (for which there are several mnemonic devices) should be properly used. For instance, students should learn that in the expression $25 + 16 \times 6$, the multiplication is performed first ($16 \times 6 = 96$), and then the addition ($25 + 96 = 121$).

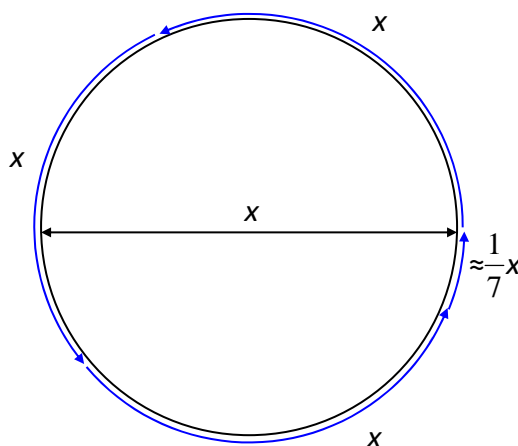
Measurement and Geometry 1.2 Know common estimates of π (3.14; 22.7) and use these values to estimate and calculate the circumference and the area of circles; compare with actual measurements.

Problem: How many segments x will fit on the circumference of the circle?



Solution: Since the circumference of this circle is πx , we see that π diameters would fit around the circumference of the circle. That is, approximately $3\frac{1}{7}$ diameters would fit around the circle.

If we had pieces of string of length x units, then we could wrap approximately $3\frac{1}{7}$ pieces around the circumference:



Further Explanation: Students can experiment with circles of differing diameters to develop an intuitive understanding that the ratio of the circumference of any circle to its diameter is a constant. This constant of course is the irrational number π .

Measurement and Geometry 1.1 Understand the concept of a constant such as π ; know the formulas for the circumference and area of a circle.

Measurement and Geometry 1.3 Know and use the formulas for the volume of triangular prisms and cylinders (area of base \times height); compare these formulas and explain the similarity between them and the formula for the volume of a rectangular solid.

Problem: Use the formula $\pi r^2 h$ for the volume of a right circular cylinder. What is the ratio of the volume of such a cylinder to the volume of one having half the height but the same radius? What is the ratio of the volume of such a cylinder to the volume of one having the same height but half the radius? (This problem also applies to Number Sense Standard 1.2.)

Solution: Notice that the equation for the volume of a cylinder, $V = \pi r^2 h$, is linear in h . Therefore, if the height of a given cylinder is multiplied by a constant such as 2, the volume should also be multiplied by 2. In symbols, if $V_1 = \pi r^2 h$ and $V_2 = \pi r^2 \frac{h}{2}$, then the ratio of V_1 to V_2 is

$$\begin{aligned}\frac{V_1}{V_2} &= \frac{\pi r^2 h}{\pi r^2 (h/2)} \\ &= \frac{\pi}{\pi} \cdot \frac{r^2}{r^2} \cdot \frac{h}{h/2} \\ &= h \div \frac{h}{2} = \frac{h}{1} \cdot \frac{2}{h},\end{aligned}$$

or, $V_1/V_2 = 2$.

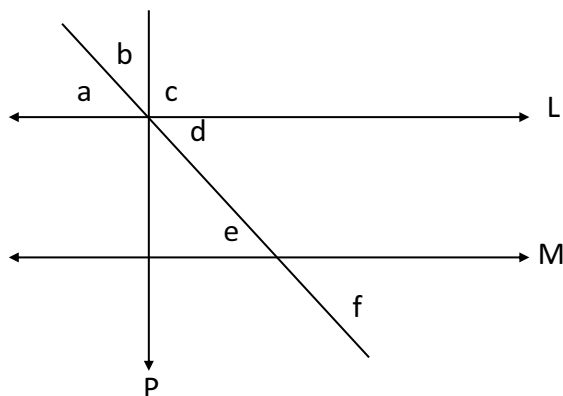
On the other hand, the formula is quadratic in r , in the sense that r is squared. Thus, if the radius is multiplied by a constant such as 2, then the volume should be multiplied by 4. In symbols, if $V_1 = \pi r^2 h$ and $V_3 = \pi (\frac{r}{2})^2 h$, then the ratio of V_1 to V_3 is

$$\begin{aligned}\frac{V_1}{V_3} &= \frac{\pi r^2 h}{\pi (r/2)^2 h} \\ &= \frac{\pi}{\pi} \cdot \frac{r^2}{r^2/4} \cdot \frac{h}{h} \\ &= r^2 \div \frac{r^2}{4} = \frac{r^2}{1} \cdot \frac{4}{r^2},\end{aligned}$$

so that $V_1/V_3 = 4$.

Measurement and Geometry 2.1 Identify angles as vertical, adjacent, complementary, or supplementary and provide descriptions of these terms.

Problem: Line L is parallel to line M . Line P is perpendicular to L and M . Name the following angles. If none can be named, leave the space blank.

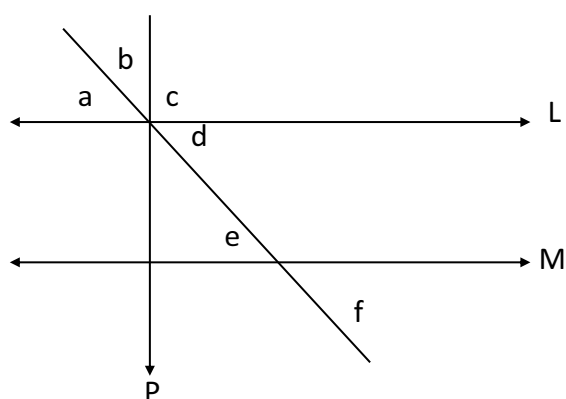


1. Complementary _____
2. Supplementary _____
3. Vertical _____
4. Alternate Interior _____
5. Corresponding _____
6. Acute _____
7. Right _____
8. Obtuse _____

Solution:

1. Complementary angles are pairs of angles whose measures sum to 90° . Such pairs include a and b , b and d (since d has the same measure as a), b and e , and b and f .
2. Supplementary angles are pairs of angles whose measures sum to 180° . There are no labeled pairs of such angles.
3. Vertical angle pairs are non-adjacent angles formed by intersecting lines. These include the pairs a and d , and e and f .

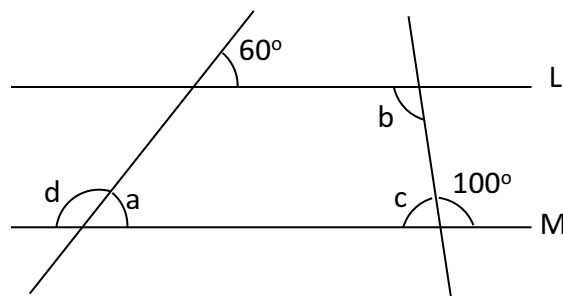
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4. Alternate interior angles are angles on opposite sides of a transversal interior to two lines cut by the transversal. These include d and e .
5. Corresponding angles are created when a transversal cuts two lines ℓ and m . They are on the same side of the transversal and the same side of ℓ as they are of m . Examples are a and e , and d and f .
6. Acute angles have measure less than 90° . These include a , b , d , e and f .
7. Right angles have measure 90° . Angle c is the only right angle.
8. Obtuse angles have measure greater than 90° . There are no obtuse angles labeled.

Measurement and Geometry 2.2 Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to solve problems involving an unknown angle.

Problem: Line L is parallel to line M . Give the number of degrees for the lettered angles.



Solution: Since L and M are parallel, pairs of corresponding angles are congruent to each other. Therefore, angle a measures 60° . Angle b measures 100° since alternate interior angles are congruent to each other. Since angle d and angle a are supplementary, the measure of angle d is $180^\circ - 60^\circ = 120^\circ$. Angle c is supplementary to the 100° angle, so angle c measures 80° .

Statistics, Data Analysis, and Probability 2.2 Identify different ways of selecting a sample (e.g., convenience sampling, responses to a survey, random sampling) and which method makes a sample more representative for a population.

Statistics, Data Analysis, and Probability 2.5 Identify claims based on statistical data and, in simple cases, evaluate the validity of the claims.

Problem: Fifty red marbles are placed in a box containing an unknown number of green marbles. The box is thoroughly mixed, and 50 marbles are taken out. Ten of those marbles are red. Does this fact imply that the number of green marbles was 200?

Solution: Since the box was thoroughly mixed, there is a high probability that the handful of marbles that was taken from the box represents the true proportion of red marbles to green. Since the handful of marbles consisted of 10 red and 40 green, this implies that ratio of red to green is 1:4. Assume this, and let x represent the number of green marbles. Then we can set up the proportion

$$\frac{1}{4} = \frac{50}{x} \quad \Rightarrow \quad x = 200.$$

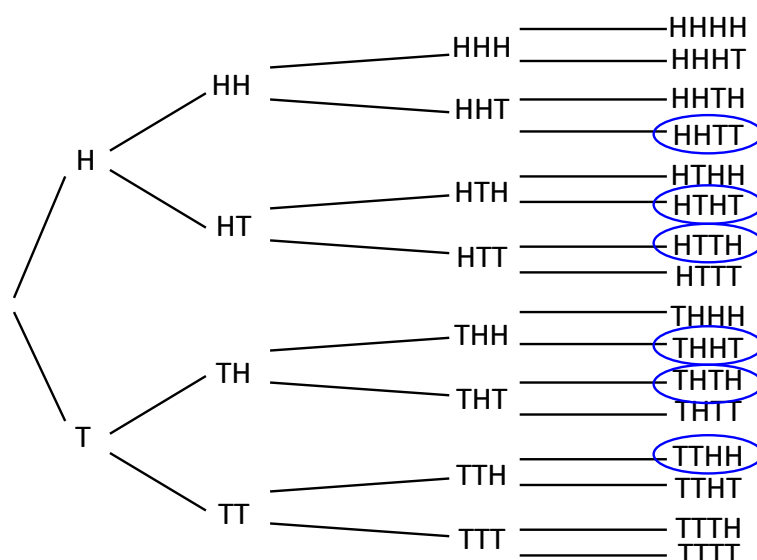
Therefore, we can make a strong argument that there are 200 green marbles in the box.

Statistics, Data Analysis, and Probability 3.1 Evaluate the reasonableness of the solution in the context of the original situation.

Statistics, Data Analysis, and Probability 3.4 Understand that the probability of either of two disjoint events occurring is the sum of the two individual probabilities and that the probability of one event following another, in independent trials, is the product of the two probabilities.

Problem: Make a tree diagram of all the possible outcomes of four successive coin tosses. How many paths in the tree represent two heads and two tails? Suppose the coin is weighted so that there is a 60% probability of heads with each coin toss. What is the probability of one head and one tail?

Solution: A tree diagram is shown below:



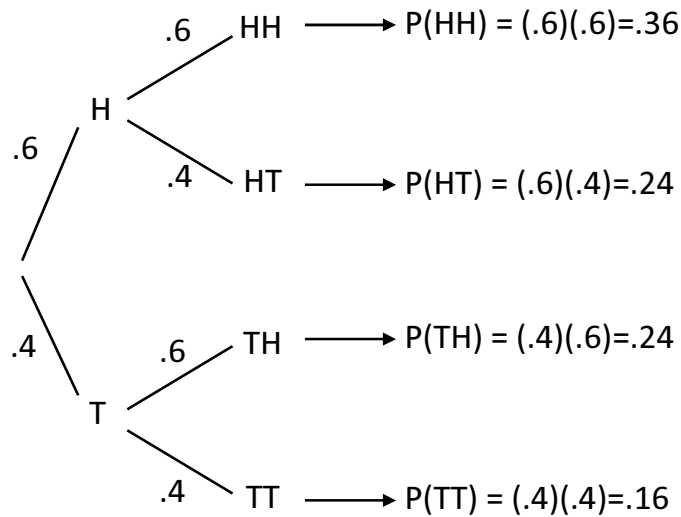
Clearly, there are 6 paths that lead to two heads and two tails.

For the second question, we assume that the weighted coin is only flipped twice. As the tree diagram above shows, there are two ways to get the outcome required: HT and TH . The probability of flipping a head is 60%, so the probability of flipping a tail is 40%. Since these are independent events, we can find the probability of flipping a head and then a tail by multiplying the probabilities. In the end, we are looking for the probability of flipping HT or TH , so we compute:

$$\begin{aligned}
 P(HT \text{ or } TH) &= P(HT) + P(TH) \\
 &= (.6)(.4) + (.4)(.6) \\
 &= .24 + .24 = .48
 \end{aligned}$$

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Often, a tree diagram with the branches labeled with the probabilities of given events is constructed, as in the picture below.



Thus, we see that $P(HT \text{ or } TH) = .48$ as found previously. Note also that

$$\begin{aligned} P(HH \text{ or } TT) &= P(HH) + P(TT) \\ &= .36 + .16 = .52 \end{aligned}$$

But this is what we would expect, since

$$P(HH \text{ or } TT) = 1 - P(HT \text{ or } TH).$$